Math 132: Differential Topology

§ Smooth manifolds

$\mathbb{R}^k$: k-dimensional Euclidean space

i.e. a point $x \in \mathbb{R}^k$ is a k-tuple $x = (x_1, \cdots, x_k)$ of real numbers.

<u>Def</u> For open sets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$,

a map $f : U \to V$ is called <u>smooth</u> (or C^∞)

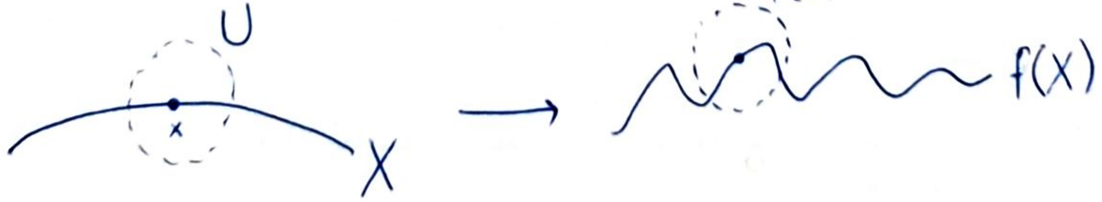
if all the partial derivatives $\dfrac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}$ exist and are continuous.

More generally, for arbitrary subsets $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^\ell$,

a map $f : X \to Y$ is called <u>smooth</u> we need an open domain to talk about partial derivatives

if for each $x \in X$ there exists an open $U \subset \mathbb{R}^k$ containing x

and a smooth map $F : U \to \mathbb{R}^\ell$ such that $F = f$ on $U \cap X$.



Prop If $f: X \to Y$ and $g: Y \to Z$ are smooth, then $g \circ f: X \to Z$ is also smooth. (easy exercise)

Def A map $f: X \to Y$ is called a diffeomorphism if it is bijective and both f and f^{-1} are smooth.

Meta Def Differential topology is the study of properties of "spaces" which are invariant under diffeomorphism.

A particularly nice class of "spaces" is:

Def A subset $M \subset \mathbb{R}^k$ is called a smooth manifold of dimension m (or simply a smooth m-manifold) if it is locally diffeomorphic to $\mathbb{R}^m$, i.e. if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open $U \subset \mathbb{R}^m$.

A diffeomorphism $\phi: U \to W \cap M$ is called a parametrization of the region $W \cap M$. The inverse diffeomorphism $\phi^{-1}: W \cap M \to U$ is called a coordinate system on $W \cap M$.

Example The unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ is a smooth 2-manifold (i.e. a smooth surface).

For instance, the diffeomorphism

$$(x,y) \longrightarrow (x,y,\sqrt{1-x^2-y^2}) \qquad (x^2+y^2<1)$$

parametrizes the region $z>0$ of S^2.

By interchanging the roles of x,y,z and changing the signs, we obtain parametrizations of regions which cover S^2.

More generally, $S^{n-1} = \{x \in \mathbb{R}^n \mid |x|=1\}$ is a smooth $(n-1)$-manifold.

Example $\{(x,y) \in \mathbb{R}^2 \mid x \neq 0,\ y = \sin(\frac{1}{x})\}$ is a smooth 1-manifold.

Rmk Note that the definition of smooth manifolds we gave is "extrinsic" in that M is embedded in $\mathbb{R}^k$.

We can also give a better, "intrinsic" definition as follows.

<u>Def</u> A Hausdorff, second countable topological space M is called a <u>topological m-manifold</u> if it is locally homeomorphic to $\mathbb{R}^m$.

<u>Def</u> A <u>coordinate chart</u> is a pair (U, x) consisting of open $U \subset M$ and a homeomorphism $x : U \to \mathbb{R}^m$ onto its image.

Two charts (U, x) and (V, y) on M are <u>smoothly related</u> if the transition function $y \circ x^{-1} : x(U \cap V) \longrightarrow y(U \cap V)$ is a diffeomorphism.

An <u>atlas</u> on M is a collection $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$ of smoothly related charts which cover M (i.e. $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M$).

A <u>smooth structure</u> on M is a maximal atlas.

(Any atlas can be completed to a unique smooth structure.)

<u>Def</u> (Intrinsic definition of smooth manifolds)

A <u>smooth m-manifold</u> is a topological m-manifold equipped with a smooth structure.